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EFFECT OF AN APERTURE ON MEASUREMENT OF THE AXIAL DISTRIBUTION FUNCTION IN A MAGNETICALLY CONFINED PLASMA

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16. Abstract

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EFFECT OF AN APERTURE ON MEASUREMENT OF THE AXIAL DISTRIBUTION FUNCTION IN A MAGNETICALLY CONFINED PLASMA

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SUMMARY

Theoretical calculations have been performed to determine the distortion in the axial velocity distribution when a magnetically confined plasma passes through an aperture. A general solution is obtained in terms of aperture length to radius ratio, transverse to axial temperature ratio, and normalized length. Results are shown for magnetic field strengths ranging from zero to infinity. It is found that the use of strong magnetic fields allows the aperture to pass a plasma sample from which the true axial distribution function may be obtained.

The effects of distortion in the axial velocity distribution on a measurement of axial temperature by taking the slope of the natural logarithm of current versus voltage is also discussed. It is shown that the error in such a measurement can generally be kept below 30.5 percent.

INTRODUCTION

The usual methods of finding the axial energy distribution of particles within a magnetically confined plasma consist of allowing a small sample of the plasma to pass through an aperture and performing an energy analysis on the resulting low-density plasma. This energy analysis may be done by using electrostatic fields alone (refs. 1 and 2) or by means of a combination of electric and magnetic fields (refs. 3 and 4). The aperture may be placed at the end of a mirror machine (refs. 1 and 4), may be attached to the body of a probe which is placed into the plasma (ref. 2), or can consist of a magnetically shielded duct (ref. 3) so that the particles are extracted transverse to the magnetic field.

Analytical solutions of the change in the distribution function of particles passing through the aperture have generally been restricted to treating the particles as having rectilinear motion (refs. 1 and 2), in which case the effects of the magnetic field are considered negligible. Another method, which also neglects magnetic fields, has been to consider the aperture as a boundary separating two regions of different electric field strength and then determining what effect these fields have on the distribution function (refs. 3 and 5).

One exception is a recent paper by Anderson, Eggleton, and Keesing (ref. 6), who treat the case of a point source of plasma in a magnetic field and show that the particle distribution is strongly affected when the plasma passes through an infinitely thin aperture.

It is the purpose of this report to extend the calculations of Anderson, Eggleton, and Keesing to the case of a uniformly distributed, magnetically confined plasma with anisotropic velocity distribution which is allowed to pass through apertures of arbitrary length and diameter. The general method of solution is followed by applications to apertures of specific length and diameter. These solutions are then used to propose criteria for aperture design which will give minimum distortion in the axial distribution function.

THEORY

Formulation of Problem

Consider a flat plate of thickness L immersed in a uniform magnetic field which is normal to its surface. Let the region to one side of this plate contain a plasma of known velocity distribution and the other side be evacuated. The plate is considered to contain a cylindrical aperture of radius R through which a portion of the plasma may flow. The situation to be considered is depicted in figure 1.

The primary goal of this report is to find the axial velocity distribution of the particles emerging from the aperture while a secondary goal is to determine whether an accurate measurement of axial plasma temperature can be obtained at the aperture exit. The variables to be considered are the aperture dimensions, the magnetic field strength, and the axial and transverse particle temperatures. The following assumptions are made:

(1) Any particle striking the bounding surfaces of the aperture is absorbed.

(2) The plasma is collisionless; that is, the mean free path is large compared with the aperture dimensions.

(3) The regions under consideration are free of electrostatic fields.

(4) Particle absorption at the walls is the only loss mechanism.

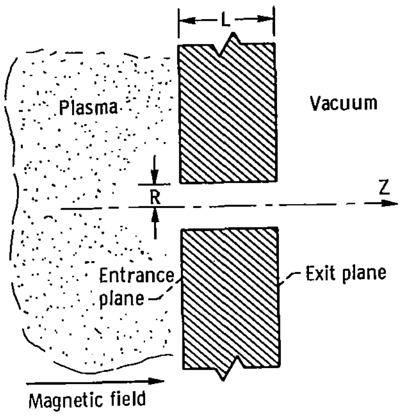


Figure 1. - Schematic of situation to be analyzed.

Because of the presence of a magnetic field, the charged particle trajectories are helices spiraling around the magnetic lines of force. It is therefore convenient to resolve this helical motion into a circular motion about the particles' guiding center superimposed on the motion of the guiding center along the direction of the magnetic field. This direction is taken as the z-axis. In the remainder of this report, the term particle orbit, or simply orbit, will denote only this circular motion projected on the entrance plane.

While the final objective is to calculate the number of particles emerging from the aperture per unit time with axial velocities between v_z and $v_z + dv_z$, the analysis will first separate the particles into two classes depending on whether the gyroradius is less than or greater than R. A transmission function for the flow of particles through the aperture will be calculated as a function of gyroradius and axial velocity for each class and the results will then be combined to compute a distribution function for the particles emerging from the aperture in terms of the initial distribution assumed.

Particles entering the plane of the aperture with gyroradii in the range $0 \leq r_g < R$ are placed in class I; while particles with gyroradii in the range $R \leq r_g < \infty$ are placed in class II. Class I particles can further be divided depending on whether their orbits are wholly within (Class Ia) or partly within (Class Ib) the aperture. The classes are summarized below.

Class	Range on r_g	Description
Ia	$0 \leq r_g < R$	Orbit completely inside of aperture
Ib	$0 \leq r_g < R$	Orbit partly intersects aperture
II	$R \leq r_g < \infty$	Orbit partly intersects aperture

The procedure followed herein will be to determine a transmission function which can then be multiplied by a velocity distribution function and integrated to obtain the velocity distribution function at the aperture exit or which can be multiplied by the velocity distribution function times the velocity and integrated to determine a particle flow at the aperture exit. The same results could be obtained by initially treating this as a flow problem and calculating particle flow directly. The particular approach using a transmission function was considered more useful.

Consider a uniform distribution of particles with a given gyroradius r_g. The location of the guiding centers of these particles on the x-y plane will also be uniformly distributed. Consider the particles within a plane slice of thickness dz such that the surface density of their guiding centers is σ. Now select an area dA in the plane of the aperture such that any particle of gyroradius r_g whose guiding center lies within dA will intersect the aperture with some or all of its orbit. The total number of such particles will be σdA. If the length of the particle orbit projecting across the aperture is S, the probability of that particle entering the aperture as it reaches the aperture entrance plane is $S/2\pi r_g$. The quantity $\sigma S dA/2\pi r_g$ then represents the total number of particles with gyroradii r_g which enter the aperture. Since the quantity S varies throughout A, this expression must be integrated. The total number of particles of gyroradii r_g which enter the aperture is thus $\sigma/2\pi r_g \int S\, dA$ with limits of integration appropriate to the value of r_g being considered. This area will be either circular or annular. In terms of the radial distance from the center of the aperture ρ, the integral becomes

$$N = \frac{\sigma}{r_g} \int_{\rho_{min}}^{\rho_{max}} S\rho \, d\rho \tag{1}$$

The limits of integration in this expression depend on the class to which the particles have been assigned.

If the particles belong to class I ($r_g < R$)

$$N = \frac{\sigma}{r_g} \int_0^{R+r_g} S\rho \, d\rho = \frac{\sigma}{r_g} \int_0^{R-r_g} 2\pi r_g \rho \, d\rho + \frac{\sigma}{r_g} \int_{R-r_g}^{R+r_g} S\rho \, d\rho \tag{2}$$

The integral whose limits are 0 to $R - r_g$ corresponds to particles belonging to class Ia while the remaining integral corresponds to particles belonging to class Ib.

If the particles belong to class II ($r_g \geq R$)

$$N = \frac{\sigma}{r_g} \int_{r_g - R}^{r_g + R} S\rho \, d\rho \tag{3}$$

Calculation of a Transmission Function

While traversing the aperture length L, each of the particles will complete a number of orbits which depend on the strength of the magnetic field, the aperture length, and the axial velocity of the particle. The orbital angle through which each particle rotates while traversing the aperture is given by

$$\theta = \frac{qBL}{m v_z} \tag{4}$$

where all the quantities used are defined in appendix A.

Therefore, the particles will travel through an arc of length $r_g\theta$ during the time it takes to pass through the aperture. Referring to figure 2, not all the particles which enter the aperture along the arc S will pass through, but only those which enter the aperture on that portion of the path given by $S - \theta r_g$. Consequently, the equations corresponding to equations (2) and (3) for the exit plane of the aperture will be given by

$$N_{thru} = \pi\sigma(R - r_g)^2 + \frac{\sigma}{r_g} \int_{R - r_g}^{\rho_{1u}} (S - \theta r_g)\rho \, d\rho \tag{5}$$

for class I ($r_g < R$) and

$$N_{thru} = \frac{\sigma}{r_g} \int_{\rho_l}^{\rho_{2u}} (S - \theta r_g)\rho \, d\rho \tag{6}$$

for class II ($r_g \geq R$). The limits have been changed from those used in equations (2) and (3) to insure that the quantity $S - \theta r_g$ will never be negative since negative values of

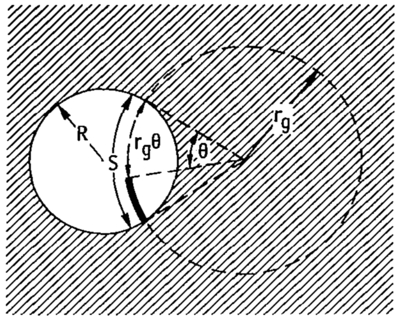


Figure 2. - Schematic depicting the portion of its orbit (denoted by heavy line) that a particle must be on in order to pass through the aperture.

the integrand represent those particles that will impinge on the aperture walls. These limits can be rigorously specified in terms of r_g and v_z, as shown in appendix B.

In the first class ($r_g < R$), the integrand would range from 2π at the lower limit to 0 at the upper limit if θ were zero. For any given value of θ, the upper limit is effectively reduced. The integral vanishes if θ exceeds 2π.

In the second class ($r_g \geq R$), the integrand is zero at both limits when θ is zero. The effect of increasing θ is to shrink both limits until at last they coalesce.

Since for any value of r_g, the uniformity of spatial distribution implies that the total number of particles which will arrive at the aperture entrance should be $\sigma\pi R^2$, a transmission function for the aperture can be defined by dividing the expressions given by equations (5) and (6) by $(\sigma\pi R^2)$ (see appendix C). This results in

$$\eta = \left(\frac{R - r_g}{R}\right)^2 + \frac{1}{\pi R^2 r_g}\int_{R-r_g}^{\rho_{1u}} (S - \theta r_g)\rho \, d\rho \qquad r_g < R \tag{7a}$$

$$= \frac{1}{\pi R^2 r_g}\int_{\rho_l}^{\rho_{2u}} (S - \theta r_g)\rho \, d\rho \qquad r_g \geq R \tag{7b}$$

In order to carry the calculations any further, we need the relation between S, ρ, and r_g. Referring to figure 3,

$$S = \varphi r_g \tag{8}$$

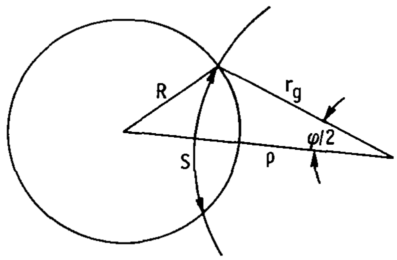


Figure 3. - Schematic representation of relation between S, ρ, and r_g.

The law of cosines gives

$$\cos \frac{\varphi}{2} = \frac{\rho^2 + r_g^2 - R^2}{2r_g\rho} \tag{9}$$

from which

$$S = 2r_g \cos^{-1}\left(a_1\rho - \frac{a_2}{\rho}\right) \tag{10}$$

where

$$a_1 = \frac{1}{2r_g} \tag{11a}$$

$$a_2 = \frac{(R^2 - r_g^2)}{2r_g} \tag{11b}$$

Hence, if we momentarily neglect the limits, the problem of finding a closed form expression for the transmission function consists of evaluating an integral of the form

$$[I]_a^b = \int_a^b \left[2 \cos^{-1}\left(a_1\rho - \frac{a_2}{\rho}\right) - \theta\right] \rho \, d\rho \tag{12}$$

As shown in appendix C, the closed form expression for this integral is given by

$$[I]_a^b = \left\{\rho^2 \cos^{-1}\left(a_1\rho - \frac{a_2}{\rho}\right) - \frac{\theta\rho^2}{2} + \frac{1 + 4a_1a_2}{4a_1^2}\sin^{-1}\left(\frac{2a_1^2\rho^2 - 1 - 2a_1a_2}{\sqrt{1 + 4a_1a_2}}\right)\right.$$

$$\left. - \frac{\left[-a_1^2\rho^4 + (1 + 2a_1a_2)\rho^2 - a_2^2\right]^{1/2}}{2a_1}\right\}\Bigg]_a^b \tag{13}$$

Hence, the transmission function becomes

$$\eta = \left(\frac{R - r_g}{R}\right)^2 + \frac{1}{\pi R^2}[I]_{R-r_g}^{\rho_{1u}} \qquad (r_g < R) \tag{14a}$$

$$= \frac{1}{\pi R^2}[I]_{\rho_l}^{\rho_{2u}} \qquad (r_g \geq R) \tag{14b}$$

Using such a transmission function, it is then simple to take a known velocity distribution, for example, a two-temperature Maxwellian, and determine its shape at the exit of the aperture.

Velocity Distribution at Aperture Exit

Let the particles in the bulk plasma have a velocity distribution function given by

$$F_i(v_x, v_y, v_z) = \frac{n_0 m}{2\pi kT_\perp}\left(\frac{m}{2\pi kT_z}\right)^{1/2} e^{m(v_x^2+v_y^2)/2kT_\perp}\, e^{-mv_z^2/2kT_z} \tag{15a}$$

$$F_i(v_\perp, v_z) = \frac{n_0 m}{kT_\perp}\left(\frac{m}{2\pi kT_z}\right)^{1/2} e^{-mv_\perp^2/2kT_\perp}\, v_\perp\, e^{-mv_z^2/2kT_z} \tag{15b}$$

The axial velocity distribution ahead of the aperture is then given by

$$F_i(v_z) = \frac{n_0 m}{kT_\perp}\left(\frac{m}{2\pi kT_z}\right)^{1/2} e^{-mv_z^2/2kT_z} \int_0^\infty e^{-mv_\perp^2/2kT_\perp} v_\perp \, dv_\perp$$

$$= n_0 \left(\frac{m}{2\pi kT_z}\right)^{1/2} e^{-mv_z^2/2kT_z} = A_1 \, e^{-mv_z^2/2kT_z} \tag{16}$$

The corresponding expression for the exit of the aperture is

$$F_f(v_z) = \frac{m}{kT_\perp} A_1 \, e^{-mv_z^2/2kT_z} \int_0^\infty e^{-mv_\perp^2/2kT_\perp} v_\perp \eta \, dv_\perp \tag{17}$$

Current at Aperture Exit

The arrival rate of particles at the aperture entrance gives rise to a current which can be expressed as

$$dI_i(v_z) = \pi R^2 q A_1 \, e^{-mv_z^2/2kT_z} v_z \, dv_z \tag{18}$$

At the aperture exit, the corresponding expression is

$$dI_f(v_z) = \frac{\pi R^2 q m A_1}{kT_\perp} e^{-mv_z^2/2kT_z} v_z \left(\int_0^\infty e^{-mv_\perp^2/2kT_\perp} v_\perp \eta \, dv_\perp\right) dv_z \tag{19}$$

If it were not for the effect of the aperture, the current could be analyzed in terms of some retarding potential which cuts off particles having velocities less than $v_{max} = 2qV/m$.

In this case equation (18) can be expressed as

$$I_i(v_z > v_{max}) = \pi R^2 q A_1 \int_{v_{max}}^{\infty} e^{-mv_z^2/2kT_z} v_z \, dv_z$$

$$= \pi R^2 q n_0 \left(\frac{kT_z}{2\pi m}\right)^{1/2} e^{-qV/kT_z} \tag{20}$$

The maximum current density occurs when $V = 0$ and is just the term in front of the exponential in equation (20).

$$I_i(v_z > v_{max}) = I_{max} \exp\left(\frac{-qV}{kT_z}\right) \tag{21}$$

From which

$$\frac{qV}{kT_z} = \ln I_{max} - \ln I_i(v_z > v_{max}) \tag{22a}$$

$$\left(\frac{d}{dV}\right) \ln I_i(v_z > v_{max}) = \frac{-q}{kT_z} \tag{22b}$$

Equation (22b) is the widely used relation which permits T_z to be determined from the slope of a retarding-potential curve. The question arises as to whether this same procedure is applicable to the aperture exit. Unfortunately, a closed-form integration is not possible for this case. One is required to resort to numerical methods to obtain a set of retarding potential curves for a range of values of R, L, $T_\perp$, T_z, and B. It appears possible to condense the requirements by a normalization process. Suppose we normalize velocities in terms of their average values, the aperture radius in terms of an average gyroradius, and aperture length in terms of the distance that a particle of average axial velocity will travel while completing one cyclotron orbit. This leads to the following normalized parameters:

$$\tilde{v}_z = \sqrt{\frac{m}{2kT_z}}\, v_z \tag{23a}$$

$$\tilde{v}_\perp = \sqrt{\frac{m}{kT_\perp}}\, v_\perp \tag{23b}$$

$$\tilde{R} = \frac{qBR}{\sqrt{mkT_\perp}} \tag{23c}$$

$$\tilde{L} = \frac{qBL}{2\pi\sqrt{2mkT_z}} \tag{23d}$$

It was also found convenient to normalize r_g and ρ as follows:

$$\tilde{r}_g = \frac{r_g}{R} \tag{24a}$$

$$\tilde{\rho} = \frac{\rho}{R} \tag{24b}$$

This permits rewriting the equations in the form

$$\tilde{I}_f = \frac{2\pi R^2 kT_z}{m}\, q \int_{\tilde{v}_{max}}^{\infty} \tilde{F}_f(v_z)\tilde{v}_z\, d\tilde{v}_z \tag{25a}$$

$$\tilde{F}_f(v_z) = n_0 \left(\frac{m}{2\pi kT_z}\right)^{1/2} e^{-\tilde{v}_z^2} \int_0^{\infty} e^{-\tilde{v}_\perp^2/2}\, \tilde{v}_\perp \tilde{\eta}\, d\tilde{v}_\perp \tag{25b}$$

$$\tilde{\eta} = (1 - \tilde{r}_g)^2 + \frac{1}{\pi}\left[\tilde{I}\right]_{1-\tilde{r}_g}^{\tilde{\rho}_{1u}} \qquad (\tilde{r}_g < 1) \tag{25c}$$

$$= \frac{1}{\pi}[\tilde{I}]_{\tilde{\rho}_l}^{\tilde{\rho}_{2u}} \qquad (\tilde{r}_g \geq 1) \tag{25d}$$

$$\tilde{I} = \left\{ \tilde{\rho}^2 \cos^{-1}\left(\frac{\tilde{\rho}^2 + \tilde{r}_g^2 - 1}{2\tilde{\rho}\tilde{r}_g}\right) - \frac{\theta\tilde{\rho}^2}{2} + \sin^{-1}\left(\frac{\tilde{\rho}^2 - \tilde{r}_g^2 - 1}{2\tilde{r}_g}\right) \right.$$

$$\left. - \left[-\frac{\tilde{\rho}^4}{4} + \left(\frac{\tilde{r}_g^2 + 1}{2}\right)\tilde{\rho}^2 - \frac{(1 - \tilde{r}_g^2)^2}{4} \right]^{1/2} \right\} \tag{25e}$$

$$\tilde{v}_{max} = \sqrt{\frac{qV}{kT_z}} \tag{25f}$$

The limits of integration on ρ can be expressed as

$$\tilde{\rho}_{1u} = \tilde{r}_g \cos\left(\frac{\theta}{2}\right) + \sqrt{1 - \tilde{r}_g^2 \sin^2\left(\frac{\theta}{2}\right)} \tag{26a}$$

$$\tilde{\rho}_{2u} = \tilde{r}_g \cos\left(\frac{\theta}{2}\right) + \sqrt{1 - \tilde{r}_g^2 \sin^2\left(\frac{\theta}{2}\right)} \tag{26b}$$

$$\tilde{\rho}_l = \tilde{r}_g \cos\left(\frac{\theta}{2}\right) - \sqrt{1 - \tilde{r}_g^2 \sin^2\left(\frac{\theta}{2}\right)} \tag{26c}$$

The following relations are useful:

$$\frac{L}{\tilde{v}_z} = \frac{\theta}{2\pi} \tag{27a}$$

$$\tilde{r}_g = \frac{\tilde{v}_\perp}{\tilde{R}} \tag{27b}$$

The use of relations (27) allows the expressions for current and the distribution function to be set up for computer integration for a range of values of $\tilde{L}$, $T_z/T_\perp$ and L/R.

Special Case B = 0

At first glance, it might appear that the case $B = 0$ is treated by simply examining equations (25) to (27) in the limit as B goes to zero. This, however, can lead to erroneous conclusions because of the extensive manipulation involving quantities which either go to zero or become unbounded when the magnetic field is set equal to zero.

We thus proceed to rederive the problem for the case when the magnetic field is absent. The similarities to the case when a magnetic field is present will be pointed out as they arise.

A particle with axial velocity v_z will traverse the aperture length in a time τ given by

$$\tau = \frac{L}{v_z} \tag{28}$$

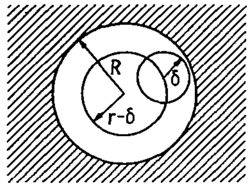
During this time, the particle will also travel a transverse distance δ, given by

$$\delta = v_\perp \tau = v_\perp \frac{L}{v_z} \tag{29}$$

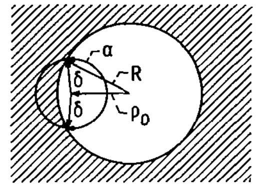
Note that the quantity δ corresponds to the arc length $r_g\theta$ for the case $B \neq 0$ and, in fact, is identical to it in value.

Suppose the position of the particle as it enters the aperture is taken as the center of a circle of radius δ. This circle defines the possible positions that every particle of velocities v_z and $v_\perp$ can have after traversing an axial distance L. (The initial position of the particle plays the role of the guiding center for the previously treated case, while the circle which defines the location of a particle at the aperture exit corresponds to the particle orbit.) Following the procedure established previously, we ask what portion of this circle of radius δ lies within the aperture. The entering particles are again broken up into two major classes depending on the value of δ, which are summarized as follows:

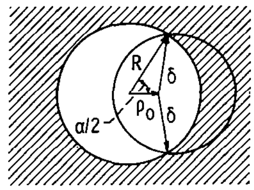
Class	Range on δ	Circle of radius δ
Ia	$0 \leq \delta < R$	Lies completely within the aperture
Ib	$0 \leq \delta < R$	Lies partly within the aperture
II	$R \leq \delta \leq 2R$	Lies partly within the aperture



(a) Class Ia; all particles enter.



(b) Class Ib; $\alpha/2\pi$ enter.



(c) Class II; $\alpha/2\pi$ enter.

Figure 4. - Representation of the fraction of particles in each class capable of leaving the aperture for the case B = 0.

Referring to figure 4, the fractions of particles capable of leaving the aperture are given by

Class	Fraction
Ia	All
Ib	$\frac{\alpha}{2\pi} = \frac{2}{\pi} \cos^{-1}\left(\frac{\rho_0^2 + \delta^2 - R^2}{2\delta\rho_0}\right)$
II	$\frac{\alpha}{2\pi} = \frac{2}{\pi} \cos^{-1}\left(\frac{\rho_0^2 + \delta^2 - R^2}{2\delta\rho_0}\right)$

A transmission function (similar to eq. (7)) can now be defined for the zero field case and is given by

$$\eta_0 = \frac{1}{R^2}\int_0^{R-\delta} 2\rho \, d\rho + \frac{2}{\pi R^2}\int_{R-\delta}^{R} \cos^{-1}\left(\frac{\rho_0^2 + \delta^2 - R^2}{2\delta\rho_0}\right)\rho_0 \, d\rho_0$$

$$= \left(\frac{1 - \delta}{R}\right)^2 + \frac{2}{\pi R^2}\int_{R-\delta}^{R} \cos^{-1}\left(\frac{\rho_0^2 + \delta^2 - R^2}{2\delta\rho_0}\right)\rho_0 \, d\rho_0 \tag{30a}$$

where $(0 \leq \delta < R)$

$$= \frac{1}{\pi R^2} \int_{\delta - R}^{R} \cos^{-1}\left(\frac{\rho_0^2 + \delta^2 - R^2}{2\delta\rho_0}\right) \rho_0 \, d\rho_0 \tag{30b}$$

where $(R \leq \delta \leq 2R)$. The integral in equation (30b) is similar to one previously used and can be expressed in closed form. This gives

$$\eta_0 = (1 - \xi)^2 + \frac{\left[\cos^{-1}\left(\frac{\xi}{2}\right) + \sin^{-1}\left(-\frac{\xi}{2}\right) - \xi\sqrt{1 - \frac{\xi^2}{4}} - \pi(1 - \xi)^2 + \frac{\pi}{2}\right]}{\pi} \tag{31a}$$

where $(0 \leq \xi < 1)$.

$$\eta_0 = \frac{\left[\cos^{-1}\left(\frac{\xi}{2}\right) + \sin^{-1}\left(-\frac{\xi}{2}\right) - \xi\sqrt{1 - \frac{\xi^2}{4}} + \frac{\pi}{2}\right]}{\pi} \tag{31b}$$

where $(1 \leq \xi \leq 2)$ and

$$\xi = \frac{\delta}{R} = L \frac{v_\perp}{R v_z} \tag{32}$$

Thus, if $0 \leq \xi \leq 2$

$$\eta_0 = \frac{\cos^{-1}}{\pi}\left(\frac{\xi}{2}\right) - \frac{\sin^{-1}}{\pi}\left(\frac{\xi}{2}\right) - \frac{\xi}{\pi}\sqrt{1 - \frac{\xi^2}{4}} + \frac{1}{2}$$

$$= \frac{2}{\pi}\cos^{-1}\left(\frac{\xi}{2}\right) - \frac{\xi}{\pi}\sqrt{1 - \frac{\xi^2}{4}} \tag{33}$$

The axial distribution function at the exit plane may now be written as

$$F_0(v_z) = n_0 \left(\frac{m}{2\pi k T_z}\right)^{1/2} e^{-\tilde{v}_z^2} \int_0^{\gamma} \eta_0 \, e^{-\tilde{v}_\perp^2/2} \, \tilde{v}_\perp \, dv_\perp \tag{34}$$

where

$$\gamma = \frac{2\sqrt{2} R\tilde{v}_z}{L}\sqrt{\frac{T_z}{T_\perp}} \tag{35}$$

The current-voltage characteristic is then found in the usual manner.

Limit as Magnetic Field Approaches Infinity

The limiting case of arbitrarily large magnetic fields may be treated by noting that the orbital angle θ always becomes greater than its maximum permissible value of 2π and the gyroradius goes to zero as the magnetic field approaches infinity. An examination of equations (25c) and (26a) reveals that the transmission function is identically equal to 1, in which case

$$\underset{\substack{\text{rear} \\ B\to 0}}{F(v_z)} = n_0 \left(\frac{m}{2\pi kT_z}\right)^{1/2} e^{-\tilde{v}_z^2} \int_0^\infty e^{-\tilde{v}_\perp^2/2}\, \tilde{v}_\perp\, d\tilde{v}_\perp$$

$$= n_0 \left(\frac{m}{2\pi kT_z}\right)^{1/2} e^{-\tilde{v}_z^2} \tag{36}$$

This equation is immediately recognizable as the axial distribution function at the entrance plane of the aperture. This is as it should be since the gyroradii of all particles are zero, and therefore the particle motion is purely along the magnetic field lines. Thus every particle that enters the aperture will reach the exit plane.

RESULTS AND DISCUSSION

Equation (25b) was set up for machine integration in terms of the normalized radius $\tilde{R}$, the length to radius ratio L/R, and the transverse to axial temperature ratio $T_\perp/T_z$.

The parameters L/R and $T_{\perp}/T_z$ were varied between values of 0.1 and 10, and $\tilde{R}$ was varied over a sufficient range so that the curves approached the values taken on at $B = 0$ and $B \to \infty$. The case $B = 0$ (eq. (34)) was also calculated, the parameter of interest being $(L/R)\sqrt{T_{\perp}/T_z}$.

The computer programs used to perform the calculations are included in appendix D.

The distribution functions were normalized by dividing through by the factor $n_0\sqrt{m/2\pi kT_z}$, and some typical results are presented in figure 5.

The effects of changing the normalized radius are presented in figure 5(a); this is equivalent to changing the magnetic field since $\tilde{R}$ is directly proportional to the strength of the magnetic field. Specifically

$$\tilde{R} = \frac{420\ 000\ B\ R}{T_{\perp}^{1/2}} \qquad \text{(electrons)} \tag{37a}$$

$$= \frac{9\ 800\ B\ R}{T_{\perp}^{1/2}} \qquad \text{(protons)} \tag{37b}$$

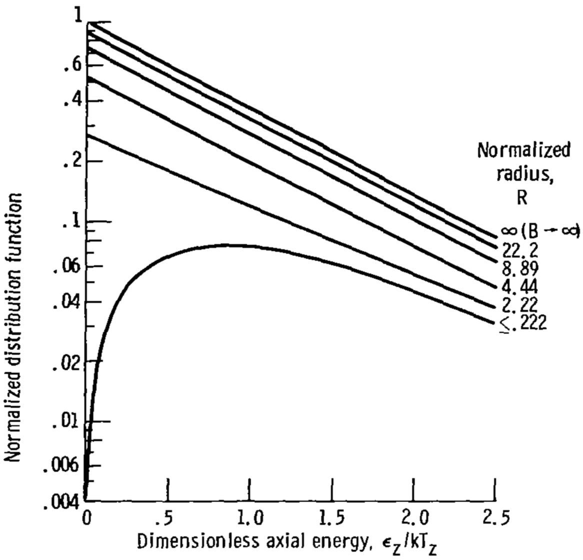
where B is given in tesla, R in meters, and $T_{\perp}$ in electron volts.

Looking further at figure 5(a) indicates that the distribution function lies near the $B \to \infty$ case when $\tilde{R} >> 10$ and lies near the $B = 0$ case when $\tilde{R} << 1.0$. In general, a large value of $\tilde{R}$ means that the aperture is much larger than the radius of gyration of most of the particles (strong field case), and thus the distribution function will remain relatively undisturbed. A small value of $\tilde{R}$ implies that the motion of the particles will be nearly rectilinear as they pass through the aperture (weak field case). In this case, the distribution function will take on the characteristics of the $B = 0$ case.

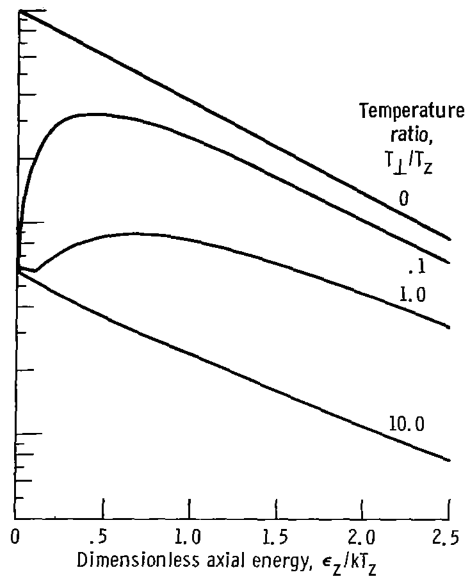
Figures 5(b) and (c) give the results of changing $T_{\perp}/T_z$ and L/R. The rather complex effects obtained by varying either of these parameters are clearly indicated.

Further insight into the effects of varying either the length to radius or the temperature ratio may be gained by looking at some limiting cases of $B = 0$. These are presented in figure 6 for various values of the combined parameter $(L/R)\sqrt{T_{\perp}/T_z}$. For a given value of this combined parameter the distribution function for nonzero magnetic fields will fall between the curve for $(L/R)\sqrt{T_{\perp}/T_z}$ equal to zero and the curve corresponding to the given value of this combined parameter.

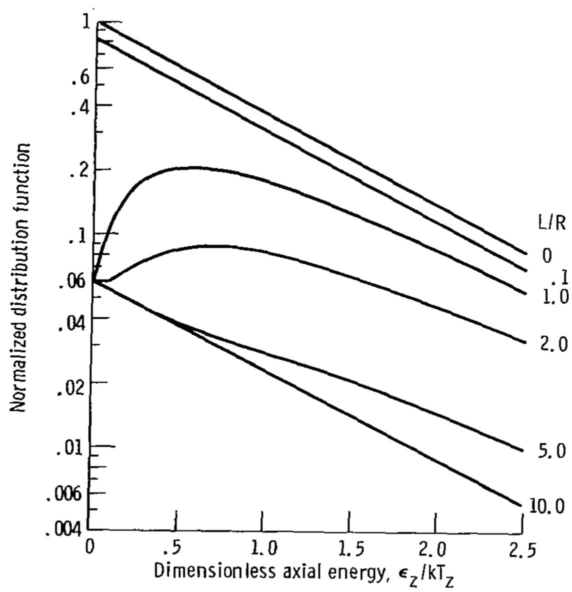
As previously mentioned, the distribution function for large values of magnetic field will be unchanged from that at the aperture entrance. Noting first the case $(L/R)\sqrt{T_{\perp}/T_z} = 0.1$, it is clear that this curve lies quite close to the undisturbed distribution function for most values of the dimensionless axial energy and departs from it only near the origin. (The case $(L/R)\sqrt{T_{\perp}/T_z} = 0$ is the undisturbed distribution function.



(a) Effect of varying the magnetic field (normalized radius). Length to radius ratio, 2.0; temperature ratio, 1.0.



(b) Effect of varying the transverse to axial temperature ratio. Length to radius ratio, 2.0; normalized aperture radius, 0.889.



(c) Effect of changing the length to radius ratio. Temperature ratio, 1.0; normalized aperture radius, 0.889.

Figure 5. - Typical distribution functions at the aperture exit showing the effects of varying the magnetic field, the temperature ratio, and the length to radius ratio.

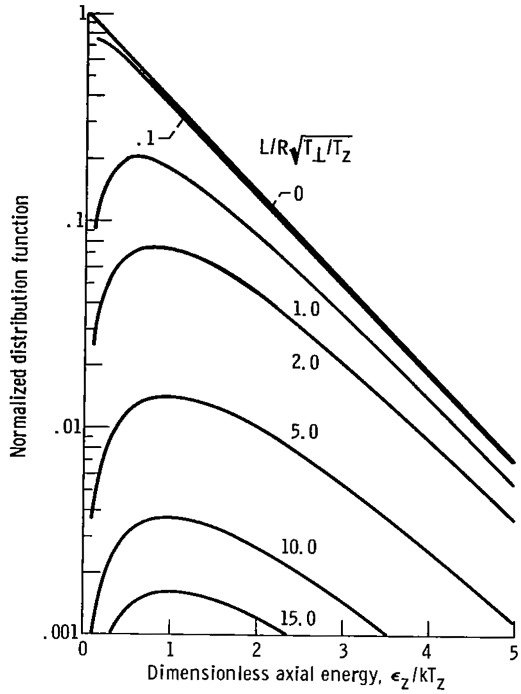


Figure 6. - Change in normalized distribution function due to variation of combined parameter $L/R\sqrt{T_\perp/T_z}$ for the case $B = 0$.

This is also the limiting case as the magnetic field approaches infinity for all values of L/R and $T_\perp/T_z$ since particles move parallel to the B-field in this case.) Looking at the other curves shows that increasing either L/R or $T_\perp/T_z$ has two effects

(1) The curve representing the distribution function moves further away from the undisturbed case.

(2) The portion of the curve which is nonlinear on a semilogarithmic plot occupies a larger range of values of dimensionless energy.

The most severe region of distortion is always near the origin and is a maximum for large values of $T_\perp/T_z$. All the distribution functions shown will reach a region of constant slope for sufficiently large values of the normalized axial energy, where a true temperature may be measured. This region may never be reached in practice. The maximum value of the normalized energy reached will depend not only on the initial particle current available but also on how quiescent the plasma in question is. Generally, it may not always be practical to measure a current versus voltage curve over a range of variation of current greater than 100. Taking these restrictions into account, temperatures were evaluated by drawing a best fit straight line through the logarithmic current versus voltage curve (on a plot of the distribution function such as fig. 6) in the region of normalized energy from 4 to 5 and using its slope to determine a temperature. Using

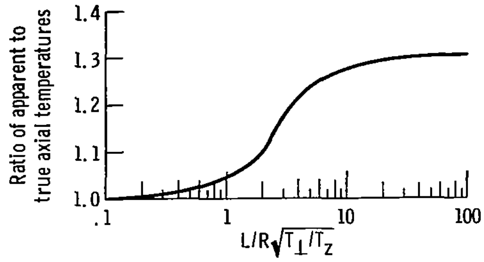


Figure 7. - Effect of the combined parameter $L/R\sqrt{T_\perp/T_z}$ on measured axial temperature. Magnetic field strength, 0.

a range of values of normalized energy closer to zero will give rise to greater errors in temperature measurement, while moving the region further away from zero will give rise to smaller errors.

Figure 7, shows the ratio of the temperature that will be indicated by a measurement to actual temperature for B = 0. The indications are that the measured temperatures will be too large unless L/R is restricted to values less than 0.1. The measured temperatures depart more and more as L/R is increased, reaching a maximum value of 1.305 times the actual temperature. Continued increases in L/R have no further effect on this measured temperature.

The variation of measured temperature with magnetic field (normalized radius) is the subject of figure 8. It is clear that increasing L/R has the effect of bringing the point where the measured temperature is equal to the true temperature nearer to the

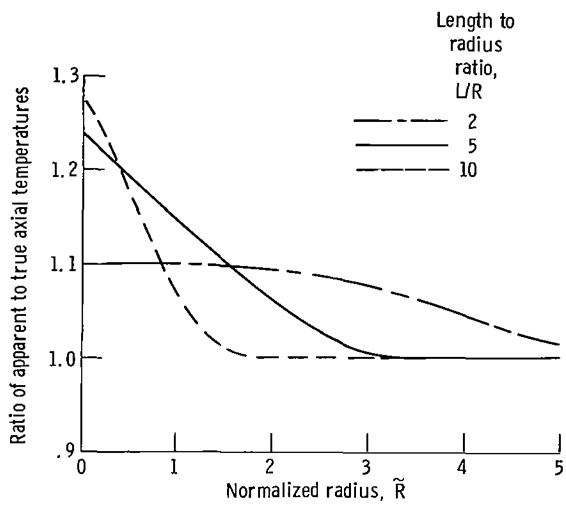


Figure 8. - Effect of normalized radius on a measurement of axial temperature. (The parameter $\tilde{R}$ is directly proportional to the strength of the magnetic field.)

origin. This indicates that comparatively small values of magnetic field will be effective in obtaining an undistorted value of the temperature for large values of L/R.

The preceeding discussion gives preliminary criteria for aperture design (i.e., either L/R must be small or the magnetic field must be chosen so that the aperture is operated in the strong field region).

In either case, it is not considered good practice to make the radius of the entrance apertures larger than the Debye length of a particle within the plasma. To establish an upper limit to collector current, consider an aperture with radius equal to a Debye length. The collected current is

$$I = q \frac{nv_{av}}{4} \pi R^2 \tag{38}$$

where v_{av} is the average axial velocity. With R equal to the Debye length, aperture current is found to be independent of density and is

$$I = 6.58\ T_z \mu A \tag{39}$$

for electrons when the axial temperature is given in electron volts.

An aperture which is built following these suggested criteria should yield a plasma efflux that truely represents the distribution function.

SUMMARY OF RESULTS

Expected changes in the axial distribution of velocities have been calculated for a collisionless plasma immersed in a magnetic field when such a plasma is allowed to flow through an aperture of finite length and diameter. The velocity distribution at the aperture exit was obtained in terms of the length to radius ratio of the aperture, the transverse to axial temperature ratio, and the normalized aperture length.

The distribution functions for selected values of the above parameters were presented along with the weak and strong magnetic field limits. The ratio of aperture length to radius and the ratio of transverse to axial temperature were varied from 0.1 to 10.0, while the magnetic field was varied from zero to infinity. The following results were noted:

1. For very strong magnetic fields, the distribution function remains undistorted regardless of the values of the other parameters.

2. Minimum distortion in the case of weak or intermediate values of magnetic field occurs when the aperture length to radius ratio is very much less than one.

3. The maximum ratio of temperature indicated at the aperture exit to that at the aperture entrance was 1.305.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, January 6, 1970,
129-02.

APPENDIX A

SYMBOLS

A_1 constant, defined in eq. (16)

dA element of area

a_1 defined in eq. (11a)

a_2 defined in eq. (11b)

B magnetic field strength

$\tilde{F}_f(v_z)$ normalized axial velocity distribution function at aperture exit

$F_i(v_x, v_y, v_z)$ velocity distribution function at aperture entrance

$F_i(v_z)$ axial velocity distribution function at aperture entrance

$F_i(v_\perp, v_z)_{front}$ velocity distribution function at aperture entrance

$F_0(v_z)$ axial velocity distribution function at aperture exit ($B = 0$)

I integral, defined in eq. (12)

$\tilde{I}$ integral, defined in eq. (25e)

$I_f(v_z)$ current at aperture exit

$I_i(v_z)$ current at aperture entrance

$I_i(v_z > v_{max})$ current at aperture entrance due to particles whose axial velocities are greater than v_{max}

I_{max} maximum current at aperture entrance

k Boltzmann constant

L aperture length

$\tilde{L}$ normalized aperture length

m particle mass

N total number of particles of gyroradius r_g entering aperture

n_0 particle number density

q electronic charge

R aperture radius

$\tilde{R}$ normalized aperture radius

r_g gyroradius

$\tilde{r}_g$ normalized gyroradius

S length of particle orbit intersecting the aperture

T_z axial temperature

$T_\perp$ transverse temperature

V retarding potential

v_x component of velocity along x-axis

v_y component of velocity along y-axis

v_z component of velocity along z-axis

$\tilde{v}_z$ normalized axial velocity

$v_\perp$ transverse velocity

$\tilde{v}_\perp$ normalized transverse velocity

γ constant, defined in eq. (35)

δ transverse distance particle travels while traversing the aperture length (B = 0)

η transmission function

$\tilde{\eta}$ normalized transmission function

η_0 transmission function (B = 0)

θ orbital angle

ξ normalized distance

ρ distance from center of aperture to guiding center of particle

$\tilde{\rho}$ normalized distance from center of aperture to guiding center of particle

ρ_l lower limit, eq. (26c)

ρ_0 distance from center of aperture to guiding center of particle (B = 0)

$\tilde{\rho}_{1u}$ upper limit, eq. (26a)

$\tilde{\rho}_{2u}$ upper limit, eq. (26b)

σ surface density of guiding centers

τ time needed for particle to traverse the aperture length

φ angle defined in fig. 3

APPENDIX B

LIMITS OF INTEGRATION

Upper and Lower Limits for Class II Particles

Figure 9 shows the upper and lower limits on ρ for a typical class II particle at various values of θ. As previously mentioned, particles with $\rho > \rho_u$ or $\rho < \rho_l$ will make the integrand $(S - r_g\theta)$ negative, and are thus absorbed by the aperture wall. Referring to figure 10 and applying the law of cosines gives

$$\cos\left(\frac{\theta}{2}\right) = \frac{r_g^2 + \rho_{u,l}^2 - R^2}{2r_g\rho_{u,l}} \tag{B1}$$

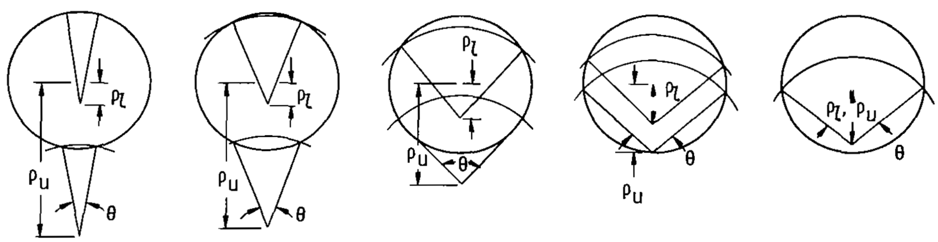
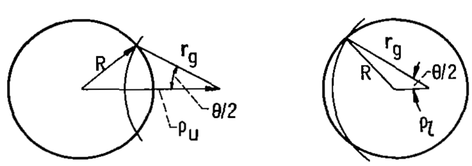


Figure 9. - Upper and lower limits for typical class II particle as function of orbital angle.



(a) Upper limit. (b) Lower limit.

Figure 10. - Upper and lower limits on distance from center of aperture to guiding center of particle for class II particles.

Solving equation (B1) for $\rho_{u,l}$ and using the notation that has been used in this report result in

$$\rho_{2u} = r_g \cos\left(\frac{\theta}{2}\right) + \sqrt{R^2 - r_g^2 \sin^2\left(\frac{\theta}{2}\right)} \tag{B2a}$$

$$\rho_l = r_g \cos\left(\frac{\theta}{2}\right) - \sqrt{R^2 - r_g^2 \sin^2\left(\frac{\theta}{2}\right)} \tag{B2b}$$

The upper and lower limits coalesce when

$$R = r_g \sin\left(\frac{\theta}{2}\right) \tag{B3}$$

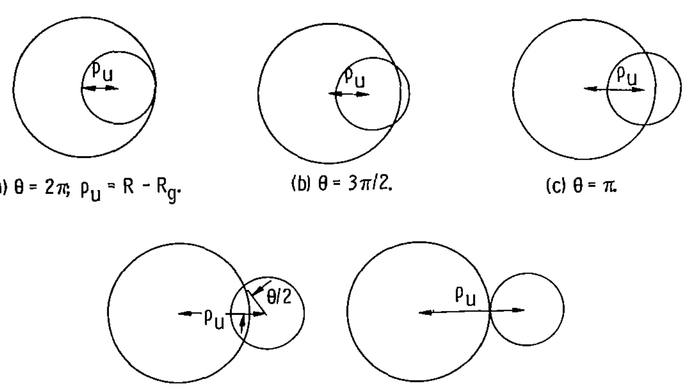
This implies that the integral vanishes for values of θ or r_g such that

$$r_g \sin\left(\frac{\theta}{2}\right) > R \tag{B4}$$

Upper Limit of Integration for Class Ib Particles

The upper limit of integration on ρ for typical class Ib particles at various values of θ are shown in figure 11. The upper limit is obtained in the same manner as the limits were obtained for class II particles and is given by

$$\rho_{1u} = r_g \cos\left(\frac{\theta}{2}\right) + \sqrt{R^2 - r_g^2 \sin^2\left(\frac{\theta}{2}\right)} \tag{B5}$$



(a) $\theta = 2\pi$; $\rho_u = R - R_g$.

(b) $\theta = 3\pi/2$.

(c) $\theta = \pi$.

(d) $\theta = \pi/2$.

(e) $\theta = 0$; $\rho_u = R + R_g$.

Figure 11. - Upper limit on distance from center of aperture to guiding center of particle ρ for typical class Ib particle at various values of orbital angle θ.

APPENDIX C

INTEGRATION OF THE TRANSMISSION FUNCTION

Integration of Equation (12)

It was previously noted that the problem of finding a closed form expression for the transmission function consists in evaluating an integral of the form

$$I = \int_a^b \left[2 \cos^{-1}\left(a_1\rho - \frac{a_2}{\rho}\right) - \theta\right] \rho \, d\rho$$

$$= \int_a^b 2 \cos^{-1}\left(a_1\rho - \frac{a_2}{\rho}\right) \rho \, d\rho - \frac{\theta\rho^2}{2}\Bigg|_a^b \tag{C1}$$

Consider the equivalent expression for the integral in equation (C1):

$$I' = \int_a^b x \cos^{-1} u \, dx \tag{C2}$$

Integrating by parts converts this equation to

$$I' = \frac{x^2}{2} \cos^{-1} u \Bigg|_a^b + \int_a^b \frac{x^2}{2\sqrt{1 - u^2}} \frac{du}{dx} dx \tag{C3}$$

Hence, the original integral becomes

$$I = \rho^2 \cos^{-1}\left(a_1\rho - \frac{a_2}{\rho}\right)\Bigg|_a^b - \frac{\theta\rho^2}{2}\Bigg|_a^b + \int_a^b \frac{\left(a_1 + \frac{a_2}{\rho^2}\right)\rho^2 \, d\rho}{\left[1 - \left(a_1\rho - \frac{a_2}{\rho}\right)^2\right]^{1/2}} \tag{C4}$$

Concerning ourselves with the remaining integral

$$\int_a^b \frac{\left(a_1 + \frac{a_2}{\rho^2}\right)\rho^2\, d\rho}{\left[1 - \left(a_1\rho - \frac{a_2}{\rho}\right)^2\right]^{1/2}} = \int_a^b \frac{\left(a_1\rho^2 + a_2\right)\rho\, d\rho}{\left[-a_1^2\rho^4 + (1 + 2a_1a_2)\rho^2 - a_2^2\right]^{1/2}}$$

$$= \frac{1}{2}\int_{a^2}^{b^2} \frac{(a_1 v + a_2)\, dv}{\sqrt{V}}$$

$$= \frac{a_1}{2}\int_{a^2}^{b^2} \frac{v\, dv}{\sqrt{V}} + \frac{a_2}{2}\int_{a^2}^{b^2} \frac{dv}{\sqrt{V}}$$

$$= \left.\frac{-\sqrt{V}}{2a_1}\right]_{a^2}^{b^2} + \left.\frac{1 + 4a_1a_2}{4a_1^2}\sin^{-1}\left(\frac{2a_1^2 v - 1 - 2a_1a_2}{\sqrt{1 + 4a_1a_2}}\right)\right|_{a^2}^{b^2} \tag{C5}$$

where

$$v = \rho^2 \tag{C6a}$$

$$V = -a_1^2v^2 + (1 + 2a_1a_2)v - a_2^2 \tag{C6b}$$

And finally

$$I = \left\{\rho^2 \cos^{-1}\left(a_1\rho + \frac{a_2}{\rho}\right) + \frac{\theta\rho^2}{2} - \frac{\left[-a_1^2\rho^4 + (1 + 4a_1a_2)\rho^2 - a_2^2\right]^{1/2}}{2a_1}\right.$$

$$\left.\left. + \frac{1 + 4a_1a_2}{4a_1^2}\sin^{-1}\left(\frac{2a_1^2\rho^2 - 1 - 2a_1a_2}{\sqrt{1 + 4a_1a_2}}\right)\right\}\right|_a^b \tag{C7}$$

Integration of Equations (5) and (6) for the Case $\theta = 0$

In deriving the transmission function, the statement was made that the number of particles incident on the aperture was represented by $\sigma\pi R^2$, independent of the value of the gyroradius. The proof follows.

Taking the case with r_g less than R, equation (5) may be written as

$$N_{thru} = \sigma\pi(R - r_g)^2 + \sigma I \Big|_{R-r_g}^{R+r_g} \tag{C8}$$

Substituting the value of I given in equation (C7)

$$N_{thru} = \sigma\pi(R - r_g)^2 + \sigma\left\{\rho^2 \cos^{-1}\left(a_1\rho - \frac{a_2}{\rho}\right) - \frac{\left[-a_1^2\rho^4 + (1 + 2a_1a_2)\rho^2 - a_2^2\right]^{1/2}}{2a_1}\right.$$

$$\left. + \frac{1 + 4a_1a_2}{4a_1^2}\sin^{-1}\left(\frac{2a_1^2\rho^2 - 1 - 2a_1a_2}{\sqrt{1 + 4a_1a_2}}\right)\right\}\Bigg|_{R-r_g}^{R+r_g} \tag{C9}$$

Substituting the values of a_1 and a_2 into equation (C9) one obtains

$$N_{thru} = \sigma\pi(R - r_g)^2 + \sigma\left\{\rho^2 \cos^{-1}\left(\frac{\rho^2 + r_g^2 - R^2}{2\rho r_g}\right) - \left[-\frac{\rho^4}{4} + (R^2 + r_g^2)\frac{\rho^2}{2} - \frac{(R^2 - r_g^2)^2}{4}\right]^{1/2}\right.$$

$$\left. + R^2 \sin^{-1}\left(\frac{\rho^2 - r_g^2 - R^2}{2r_gR}\right)\right\}\Bigg|_{R-r_g}^{R+r_g}$$

$$= \sigma\pi(R - r_g)^2 + \sigma\left[(R + r_g)^2 \cos^{-1}(1) - (R - r_g)^2 \cos^{-1}(-1)\right.$$

$$\left. - 0 + 0 + R^2 \sin^{-1}(1) - R^2 \sin^{-1}(-1)\right]$$

$$= \sigma\pi(R - r_g)^2 + \sigma\left[-\pi(R - r_g)^2 + \frac{\pi R^2}{2} + \frac{\pi R^2}{2}\right]$$

$$= \sigma\pi R^2 \tag{C10}$$

For the case where r_g is greater than R, equation (6) becomes

$$N_{thru} = \sigma I \Big|_{r_g - R}^{R_g + R}$$

$$= \sigma\left[(R + r_g)^2 \cos^{-1}(1) - (r_g - R)^2 \cos^{-1}(1) - 0 + 0 + R^2 \sin^{-1}(1) - R^2 \sin^{-1}(-1)\right]$$

$$= \sigma\pi R^2$$

Thus for both ranges of r_g, the total number of particles impinging on the front surface of the aperture comes out the expected value of $\sigma\pi R^2$ even when expressed in this complex form.

APPENDIX D

COMPUTER PROGRAMS

The programs that follow are written in standard FORTRAN IV language and do not require any special functions other than those normally supplied.

Zero Magnetic Field

A flow diagram for the program which calculates the distribution function for the zero magnetic field case (APERTO) is given in figure 12. This program varies the

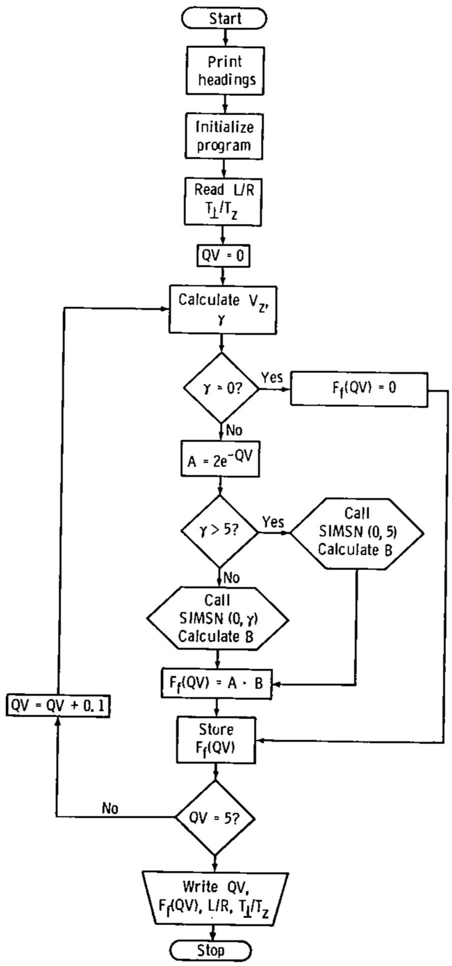


Figure 12. - Flow diagram - program APERTO.

normalized axial energy (denoted by QV within the program) from 0 to 5 in steps of 0.1 and computes the aperture exit distribution function (eq. (34)) for each value of QV. The integration over the transverse distribution function is performed by subroutine SIMSN, which is a straightforward Simpson's 1/3 rule numerical integration. Although the limits are supposed to vary from 0 to ∞, the upper limit is arbitrarily restricted to a maximum value of 5 (exp(-25) is after all a rather small number).

A flow diagram for subroutine SIMSN is given in figure 13. This subroutine is designed to repeatedly double the number of steps used in the integration until two successive values of the integral agree to within a specified number of digits.

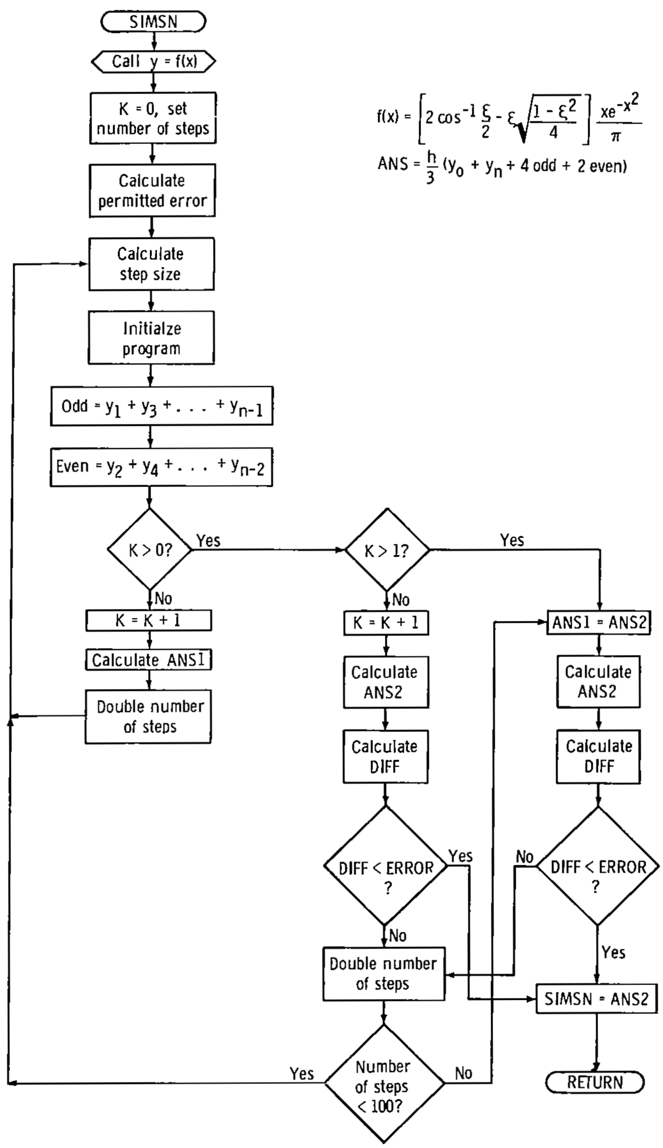


Figure 13. - Flow diagram -

The complete program listings follow.

```
C
C     COMPUTATION OF THE DISTRIBUTION FUNCTION FOR THE EFFECT OF AN
C     APERTURE ON A TWO TEMPERATURE MAXWELLIAN DISTRIBUTION WHEN B=0.
C
C
      KK=0
C
C     PRINT OUT HEADINGS.
C
 10   WRITE(6,610)
 610  FORMAT(1H1,10X,28HEFFECT OF APERTURE WHEN B=0.///)
      WRITE(6,600)
 600  FORMAT(1H ,10X,2HQV,6X,9HDIST. FN.,6X,3HL/R,6X,5HTP/TZ///)
C
C     INITIALIZE PROGRAM AND SUPPLY REQUIRED CONSTANTS.
C
      COMMON GAMA,PI
      PI=3.141593
C
C     SUPPLY INPUT DATA.
C
C     XLR IS THE APERTURE LENGTH TO RADIUS RATIO.
C     TPZ IS THE TRANSVERSE TO AXIAL TEMPERATURE RATIO.
C
      READ(5,500)XLR,TPZ
 500  FORMAT(2F10.3)
C
C     END OF INPUT DATA. START OF CALCULATION OF DISTRIBUTION
C     FUNCTION.
C
      DIMENSION DISTFN( 51),QVV( 51)
      DO 200 J=1, 51
      XJ=J-1
      QV=XJ/10.
      QVV(J)=QV
      VZ=SQRT(QV)
      GAMA=2.828428*VZ/(SQRT(TPZ)*XLR)
      IF(GAMA .EQ. 0.0) GO TO 210
      GO TO 220
 210  DISTFN(J)=0.0
      GO TO 200
 220  A=EXP(-QV)
      IF(A .LT. 1.0 E-30) A=0.0
      EXTERNAL ARG
      DIFF=0.0
      IF(GAMA .LE. 5.) B=SIMSN(0.0,GAMA,4,ARG,DIFF)
      IF(GAMA .GT. 5.) B=SIMSN(0.0,5.,4,ARG,DIFF)
      IF(B .LT. 1.0 E-30) B=0.0
      DISTFN(J)=A*B
 200  CONTINUE
      DO 300 N=1,51
      WRITE(6,605)QVV(N),DISTFN(N),XLR,TPZ
 605  FORMAT(1H ,F13.2,E15.4,2F10.2)
 300  CONTINUE
      KK=KK+1
      IF(KK .LT. 99) GO TO 10
      STOP
      END
```

```
$IBFTC SUB1

C
C     THIS FUNCTION IS THE TRANSVERSE DISTRIBUTION FUNCTION TIMES ETA.
C
      FUNCTION ARG(Y)
      COMMON GAMA,PI
      X=2.*Y/GAMA
      ETA=(2.*ARCOS(X/2.)-X*SQRT(1.-.25*X*X))/PI
      ARG=Y*ETA*EXP(-Y*Y/2.)
      RETURN
      END
```

```
C
C       FUNCTION SIMSN(A,B,N,FN,DIFF)
C
C      SIMPSON 1/3 RULE INTEGRATION.
C    A IS THE LOWER LIMIT, B THE UPPER LIMIT , N GIVES THE NUMBER OF
C    SIGNIFICANT FIGURES THE ANSWER CAN BE EXPECTED TO BE CORRECT TO,
C    AND 100*DIFF IS THE PERCENT DIFFERENCE BETWEEN THE LAST TWO
C    COMPUTED VALUES OF THE INTEGRAL.
C
C    THE CALL SEQUENCE IS
C
C       EXTERNAL FN
C       DIFF=0.0
C       ANS=SIMSN(A,B,N,FN,DIFF)
C
      FUNCTION SIMSN(A,B,N,FN,DIFF)
      EXTERNAL FN
      K=0
      HN=10.
      ERROR=1.
      DO 100 J=1,N
 100  ERROR=ERROR/10.
 200  H=(B-A)/HN
      EVEN=0.0
      ODD=0.0
      NA=HN-1.
      NB=HN-2.
      DO 300 KA=1,NA,2
      XKA=KA
      X=A+XKA*H
 300  ODD=ODD+FN(X)
      DO 400 KB=2,NB,2
      XKB=KB
      X=A+XKB*H
 400  EVEN=EVEN+FN(X)
      IF(K .GT. 0) GO TO 500
      K=K+1
      ANS1=H*(FN(A)+FN(B)+4.*ODD+2.*EVEN)/3.
      HN=2.*HN
      GO TO 200
 500  IF(K .GT. 1) GO TO 600
      K=K+1
      ANS2=H*(FN(A)+FN(B)+4.*ODD+2.*EVEN)/3.
      IF(ANS2 .EQ. 0.0) GO TO 700
      DIFF=ABS((ANS1-ANS2)/ANS2)
      IF(DIFF .LT. ERROR) GO TO 700
 550  HN=2.*HN
      IF(HN .LT. 100.) GO TO 200
 600  ANS1=ANS2
      ANS2=H*(FN(A)+FN(B)+4.*ODD+2.*EVEN)/3.
      DIFF=ABS((ANS1-ANS2)/ANS2)
      IF(DIFF .LT. ERROR) GO TO 700
      GO TO 550
 700  SIMSN=ANS2
      RETURN
      END
```

Nonzero Magnetic Field

The equations used in evaluating the distribution function for the case of a finite magnetic field are (25b) to (25e), (26) and (27). The flow diagram for the main program (APERT) is given in figure 14. The calculation of the distribution function proceeds via subroutines FNI and FNA.

Subroutine FNI solves the relation

$$FNI = (1 - \tilde{r}_g)^2 e^{-\tilde{v}_\perp^2/2} \tilde{v}_\perp \tag{D1}$$

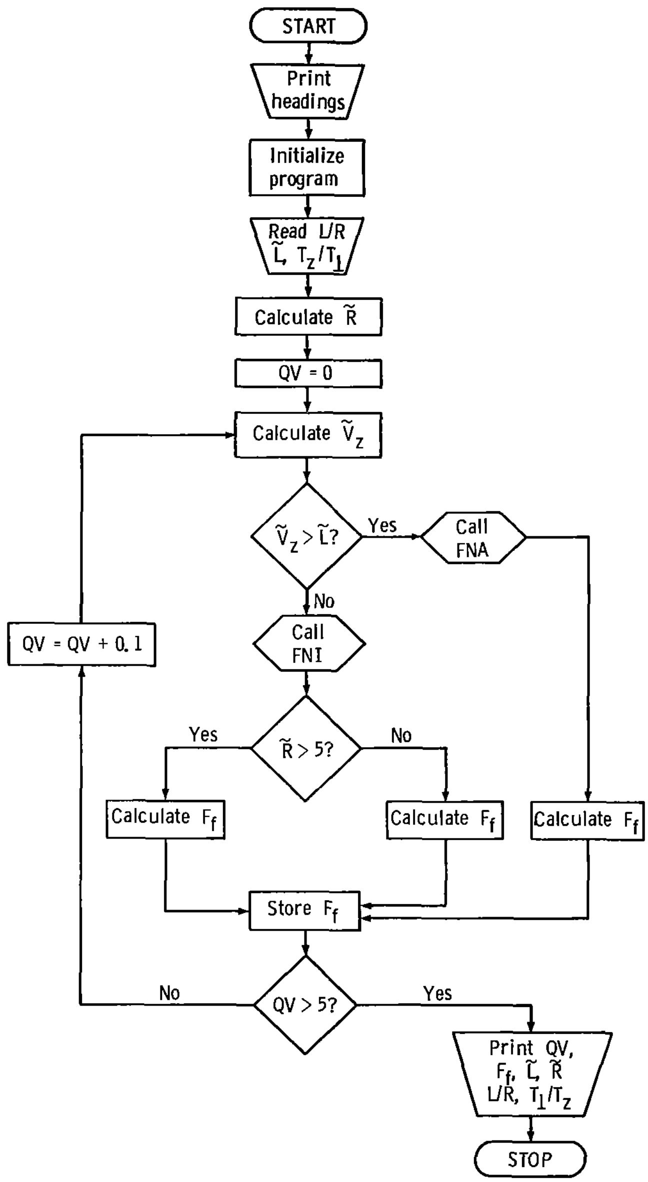


Figure 14. - Flow diagram - program APERT.

and is appropriate when the value of v_z is such that only class Ia particles can pass through the aperture.

Subroutine FNA calculates eta times the transverse distribution function for the general case and is depicted in figure 15.

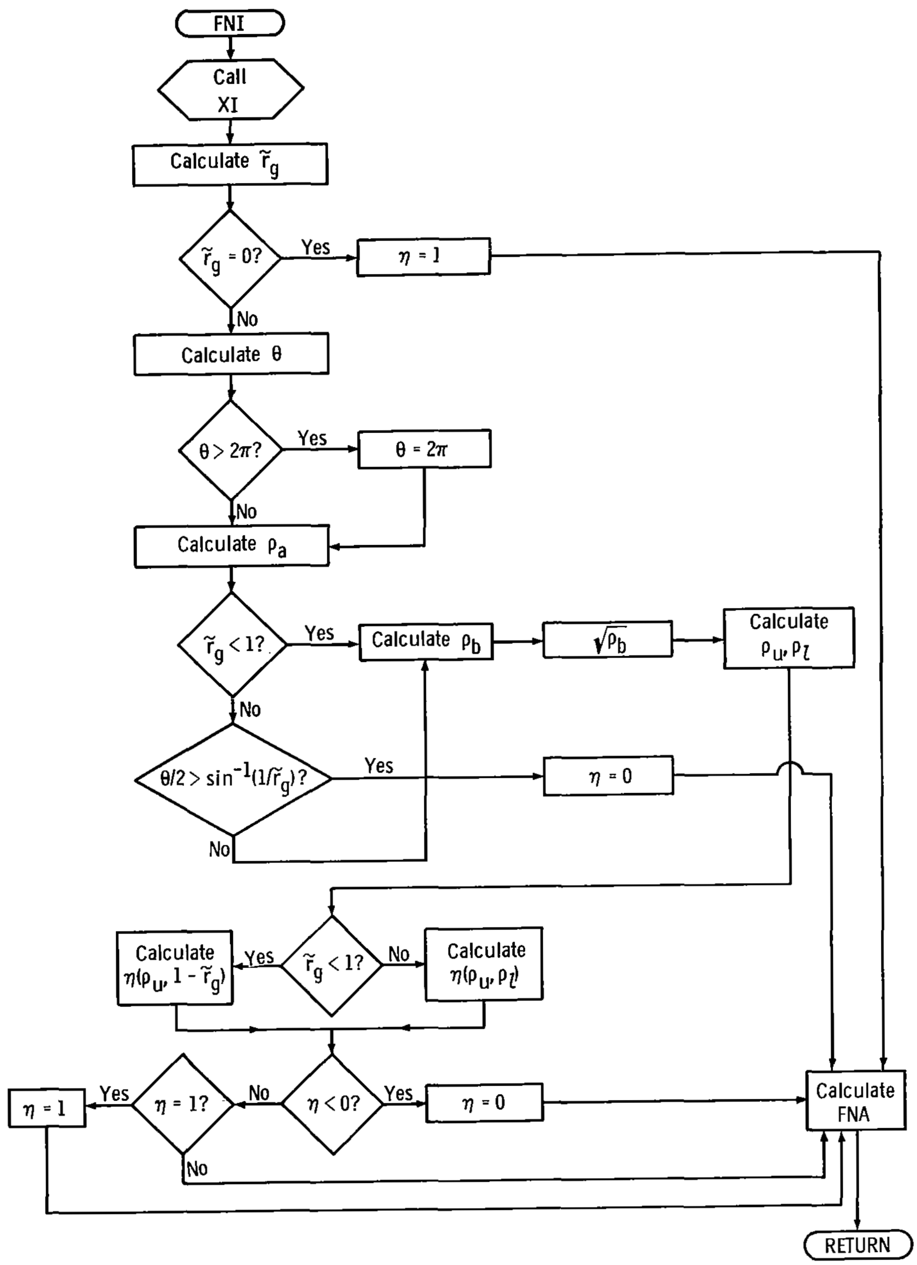


Figure 15. - Flow diagram subroutine FNA.

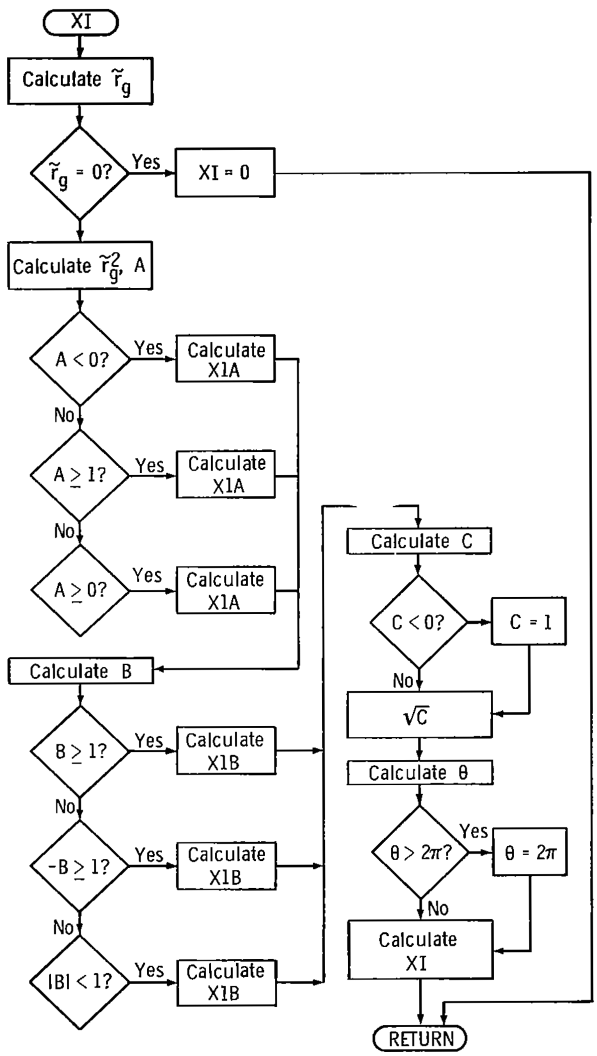


Figure 16. - Flow diagram subroutine XI.

Figure 16 is a flow diagram for subroutine XI, which is a solution of the integral used in obtaining eta (eq. (25e)). The program listings follow.

```
C
C     COMPUTATION OF THE EFFECT OF AN APERTURE ON A TWO TEMPERATURE
C    MAXWELLIAN DISTRIBUTION FUNCTION.
C
C    A MAGNETIC FIELD IS ASSUMED PERPENDICULAR TO THE APERTURE PLANE.
C
C    THE NORMALIZED APERTURE LENGTH=47200*B*L/SQRT(TZ)
C    B IS THE MAGNETIC FIELD IN TESLA.
C    L IS THE APERTURE LENGTH IN METERS.
C    TZ IS THE AXIAL TEMPERATURE IN E.V.
C
C    PRINT OUT HEADINGS AND SET COUNTER.
C
         KK=0
 10      WRITE(6,610)
 610     FORMAT(1H1,20X,43HEFFECT OF APERTURE ON DISTRIBUTION FUNCTION///)
         WRITE(6,600)
 600  0  FORMAT(1H ,5X,2HQV,6X,9HDIST. FN.,6X,3HL/R,11X,2HLN,12X,2HRN,
     1   12X,5HTP/TZ///)
C
C    INITIALIZE PROGRAM AND SUPPLY REQUIRED CONSTANTS.
C
         REAL LN,LR
         COMMON LN,LR,TZTP,PI,RN,VZ
         PI=3.141593
         KK=KK+1
C
C    INPUT DATA FOLLOWS.
C      LR IS THE LENGTH TO RADIUS RATIO OF THE APERTURE.
C      LN IS THE NORMALIZED APERTURE LENGTH.
C      TZTP IS THE RATIO OF THE AXIAL AND TRANSVERSE TEMPERATURES.
C
         READ(5,500)LR,LN,TZTP
 500     FORMAT(3F10.4)
         TPTZ=1./TZTP
C
C    END OF INPUT DATA, START OF CALCULATION OF DISTRIBUTION
C    FUNCTION.
C      RN IS THE NORMALIZED APERTURE RADIUS.
C
         RN=2.*PI*LN*SQRT(2.*TZTP)/LR
         DIMENSION DISTFN(51),QVV(51)
         DO 200 J=1,51
         XJ=J-1
         QV=XJ/10.
         QVV(J)=QV
         VZ=SQRT(QV)
C
C    CHECK TO SEE IS VZ IS GT THAN MIN VALUE RORD.
C
         IF(VZ .GT. LN) GO TO 100
         LRN=RN
         EXTERNAL FNI
         DIFF=0.0
         IF(RN .LT. 5.) DISTFN(J)=EXP(-QV)*SIMSN(0.0,RN,4,FNI,DIFF)
         IF(RN .GE. 5.) DISTFN(J)=EXP(-QV)*SIMSN(0.0,5.,4,FNI,DIFF)
         GO TO 200
 100     CONTINUE
         EXTERNAL FNA
         DIFF=0.0
         DISTFN(J)=EXP(-QV)*SIMSN(0.0,5.,5,FNA,DIFF)
 200     CONTINUE
         DO 300 N=1,51
         WRITE(6,605) QVV(N),DISTFN(N),LR,LN,RN,TPTZ
 605     FORMAT(1H ,F8.4,5E14.4)
 300     CONTINUE
         IF(KK .LT. 175) GO TO 10
         STOP
         END
```

```
C
C     THIS FUNCTION IS THE TRANSVERSE DISTRIBUTION FUNCTION TIMES
C     ETA FOR CLASS 1A PARTICLES.
C
          FUNCTION FNI(VT)
          COMMON LN,LR,TZTP,PI,RN,VZ
          REAL LN,LR
          IF(VT .GT. 7.0) GO TO 10
          GO TO 20
   10     FNI=0.0
          GO TO 100
   20     RG=VT/RN
          FNI=VT*(1.-RG)*(1.-RG)*EXP(-VT*VT/2.)
  100     CONTINUE
          RETURN
          END
```

```
C
C         FUNCTION SIMSN(A,B,N,FN,DIFF)
C
C        SIMPSON 1/3 RULE INTEGRATION.
C     A IS THE LOWER LIMIT, B THE UPPER LIMIT , N GIVES THE NUMBER OF
C     SIGNIFICANT FIGURES THE ANSWER CAN BE EXPECTED TO BE CORRECT TO,
C     AND 100*DIFF IS THE PERCENT DIFFERENCE BETWEEN THE LAST TWO
C     COMPUTED VALUES OF THE INTEGRAL.
C
C     THE CALL SEQUENCE IS
C
C         EXTERNAL FN
C         DIFF=0.0
C         ANS=SIMSN(A,B,N,FN,DIFF)
C
          FUNCTION SIMSN(A,B,N,FN,DIFF)
          EXTERNAL FN
          K=0
          HN=10.
          ERROR=1.
          DO 100 J=1,N
  100     ERROR=ERROR/10.
  200     H=(B-A)/HN
          EVEN=0.0
          ODD=0.0
          NA=HN-1.
          NB=HN-2.
          DO 300 KA=1,NA,2
          XKA=KA
          X=A+XKA*H
  300     ODD=ODD+FN(X)
          DO 400 KB=2,NB,2
          XKB=KB
          X=A+XKB*H
  400     EVEN=EVEN+FN(X)
          IF(K .GT. 0) GO TO 500
          K=K+1
          ANS1=H*(FN(A)+FN(B)+4.*ODD+2.*EVEN)/3.
          HN=2.*HN
          GO TO 200
  500     IF(K .GT. 1) GO TO 600
          K=K+1
          ANS2=H*(FN(A)+FN(B)+4.*ODD+2.*EVEN)/3.
          DIFF=ABS((ANS1-ANS2)/ANS2)
          IF(DIFF .LT. ERROR) GO TO 700
  550     HN=2.*HN
          IF(HN .LT. 100.) GO TO 200
  600     ANS1=ANS2
          ANS2=H*(FN(A)+FN(B)+4.*ODD+2.*EVEN)/3.
          DIFF=ABS((ANS1-ANS2)/ANS2)
          IF(DIFF .LT. ERROR) GO TO 700
          GO TO 550
  700     SIMSN=ANS2
          RETURN
          END
```

```
C
C     THIS FUNCTION IS THE TRANSVERSE DISTRIBUTION FUNCTION TIMES
C     ETA FOR ALL PARTICLES WHOSE AXIAL VELOCITY EXCEEDS THE MINIMUM
C     REQUIRED.
C
C     ROU IS THE UPPER LIMIT OF INTEGRATION.
C     ROL IS THE LOWER LIMIT OF INTEGRATION WHEN RG .GE. 1.
C
      FUNCTION FNA(VT)
      COMMON LN,LR,TZTP,PI,RN,VZ
      REAL LN,LR
      EXTERNAL XI
      RG=VT/RN
      IF(RG .EQ. 0.0) GO TO 35
      GO TO 40
 35   ETA=1.
      GO TO 100
 40   THETA=2.*PI*LN/VZ
      IF(THETA .GT. 2.*PI) THETA=2.*PI
 25   ROA=RG*COS(THETA/2.)
      IF(RG .LT. 1.) GO TO 30
      IF(THETA .GT. 2.*ARSIN(1./RG)) GO TO 10
 30   ROB=1.-RG*RG*SIN(THETA/2.)*SIN(THETA/2.)
      IF(ROB .LE. 0.0) GO TO 10
      GO TO 20
 10   ETA=0.0
      GO TO 100
 20   ROBA=SQRT(ROB)
      ROU=ROA+ROBA
      ROL=ROA-ROBA
     0IF(RG .LT. 1.) ETA=XI(VT,ROU,RN)/PI+.5
     1   +THETA*(1.-RG)*(1.-RG)/(2.*PI)
      IF(RG .GE. 1.) ETA=(XI(VT,ROU,RN)-XI(VT,ROL,RN))/PI
      IF(ETA .LT. 0.0) ETA=0.0
      IF(ETA .GT. 1.) ETA=1.
 100  FNA=VT*ETA*EXP(-VT*VT/2.)
      RETURN
      END
```

```
      FUNCTION XI(VT,RO,RN)
C
C     THIS FUNCTION GIVES THE VALUE OF THE INTEGRAL USED TO CALCULATE ETA
C     VT IS THE NORMALIZED TRANSVERSE VELOCITY, RO THE LIMIT OF
C     INTEGRATION, THETA THE ORBITAL ANGLE, AND RN THE NORMALIZED
C     APERTURE RADIUS.
C
      COMMON LN,LR,TZTP,PI,RN,VZ
      REAL LN,LR
      RG=VT/RN
      IF(RG .EQ. 0.0) GO TO 50
      RGS=RG*RG
      A=(RO*RO-RGS-1.)/(2.*RG)
      IF(A .LT. 0.0) XIA=-ARSIN(-A)
      IF(A .GE. 1.) XIA=PI/2.
      IF(A .GE. 0.0) XIA=ARSIN(A)
 70   B=(RO*RO+RGS-1.)/(2.*RO*RG)
      IF(B .GE. 1.) XIB=0.0
      IF(-B .GE. 1.) XIB=RO*RO*PI
      IF(ABS(B) .LT. 1.) XIB=RO*RO*ARCOS(B)
 45   C=-RO**4/4.+RO*RO*(RGS+1.)/2.-(1.-RGS)**2/4.
      IF(C .LT. 0.0) C=0.0
      XIC=SQRT(C)
      THETA=2.*PI*LN/VZ
      IF(THETA .GT. 2.*PI) THETA=2.*PI
 16   XI=XIA+XIB-XIC-THETA*RO*RO/2.
      GO TO 60
 50   XI=0.0
 60   CONTINUE
      RETURN
      END
```

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